MARKETING SCIENCE
Vol. 7, No. 2, Spring 1988
Printed in U.S.A.

OPTIMAL PRICE AND ADVERTISING POLICY FOR A CONVENIENCE GOODS RETAILER

GUSTAV FEICHTINGER, ALFRED LUHMER AND GERHARD SORGER
University of Technology, Vienna
University of Bielefeld
University of Technology, Vienna

Convenience goods are bought without much consideration or effort on the part of the consumer. Buyers don't try to get the best bargain when purchasing individual convenience items. Instead, they adapt their store choice habits so that they can expect, on average, good value for money in the long run. Store choice is governed by aggregate information which is received on the price levels of stores. This is embodied in store price images. A store price image is defined in the subsequent paper as a perceived difference between the store's price level and a reference level that consumers expect of stores about which they have no particular information. Store price images are learnt adaptively over time. We propose a theoretical model of store price image adaptation and a two-stage sales response model. According to the latter, a store's price image and advertising determine the number of customers attracted per period. Sales per visit, however, depend only on the prices actually observed in the store. For this sales model, we have studied the optimal price and advertising policy making simple assumptions of cost. Our main interest is the interaction between optimal advertising and price image. It turns out that image improvement is achieved mainly through pricing; increased advertising follows only after the price image has been improved. Therefore, as long as the price image is poor relative to some optimal equilibrium level, a larger portion of the marketing budget should be allocated to lower prices rather than to advertising. In phases where the price image declines towards the equilibrium, however, heavier advertising is profitable.

(Marketing Mix; Dynamic Models; Convenience Goods; Store Price Image; Imperfect Information)

1. Introduction

This paper analyzes some aspects of the pricing and advertising policy of a retailer that deals mainly in convenience goods, e.g. a supermarket. Convenience goods are purchased by consumers without much stress on comparison and choice, some buy impulsively, some on a regular basis. Every household buys a large variety of such goods which are offered by a number of different retailers. Prices are subject to frequent change, due to special promotions. So consumers are not perfectly informed on all of the relevant prices. Therefore, purchase decisions on convenience goods are typically made in two stages. First, one decides which store to visit, and then once in that store, on which items to buy. While the latter decision can be based on observed prices, the first one will rely on past experience, advertising, and word of mouth information. Experience is based on price observations from past purchases. Single observations will

0732-2399/88/0702/0187$01.25

not usually be remembered for long but instead they will be amalgamated into a global idea of the respective store's price level or a store price image. The motivation for consumers to resort to store price images as a basis for purchasing decisions is to reduce the cognitive difficulty of remembering a large number of prices.

The price image of a supermarket will converge towards its actual price level but convergence will be rather slow for several reasons. Customers drawn by a favorable price image will not usually leave the store once and for all when they find prices not as low as expected. As a rule it will take several visits to the store until a difference between image and reality will be noticed. Because involvement in purchasing convenience goods is low, only a portion of all of the relevant prices will be perceived during one visit, and some price fluctuation is customary. Quite often a customer delays price image adaptation until (s)he happens to buy elsewhere for some extraordinary reason. Consequently, a favorable image once prevailing is an intangible asset, as is advertising goodwill in the sense of Nerlove and Arrow (1962). A company may invest in its image by offering low prices (with an appropriate advertising back up), or utilize it by raising prices selectively, and the question naturally arises, what amount of that asset will be worth maintaining for a supermarket. We will show that there exists an optimal equilibrium level for the price image. This level may change over time subject to changes in the environment of the company. So, because of unforeseeable fluctuations in the environment, the actual image will sometimes be below, sometimes above optimum. Unfortunately, it is hard to determine quantitatively which of the two situations actually prevails.

Let us assume, however, that management can somehow identify whether the store price image is above or below optimum. Then they will want to know how to respond optimally to the assessed situation. Specifically, an optimal response will require optimal tuning of marketing-mix interaction effects. This is the subject of the subsequent analysis. The proposed model is not intended to be calibrated on data. It aims only at structural insights into the optimal policy. As a primary result a dynamic counterpart of the well-known "consistency rule" turns out to apply: lower prices, if needed to enhance the image, should accompany a relatively low level of advertising. Heavy advertising is profitable only after the image has been improved. The reason is that customers do not follow the appeal of advertising mechanically. Instead they will be influenced by past experience. Low prices for advertised items will be more readily accepted as a signal of a low overall price level, if supported by a favorable price image. So, a bad image will blunt advertising.

The marketing-mix considered here consists only of prices and advertising. Quality, variety, availability, service, and display are assumed constant in time for all competing stores. Advertising, however, is understood in a broad sense to cover quite a range of instruments. We consider the size and outer appearance of outlets, accessibility, and parking space to be advertising instruments as are TV-spots, showcards, leaflets, or insertions in newspapers. Anything that draws prospects into the store and is under the immediate control of the company figures as advertising here. We do not focus on advertising content assuming that content is chosen so that pricing decisions get an optimal backup. But we will have to consider price advertising and its specific mode of action, since most of the advertising by supermarkets is price advertising.

To simplify matters we restrict our model to a situation of imperfect competition. Oligopolistic interdependencies are disregarded. This means that companies may well have to react to competitors but they do not feel that competitors react to them. One can think of two different competitive situations of this kind: first, a company may face some range of alternative actions that do not provoke competitive reaction. Such a monopolistic range, albeit small, exists quite often. It is due to habitual behavior, store loyalty, or locational and other preferences of customers. Second, a company may adopt the role of a follower in a Stackelberg game. In this situation only the leader's policy accounts for competitors' reactions. It assumes that competitors optimize their reactions, given the leader's action. So, in order to solve the leader's problem, the followers' optimum is needed as a function of the leader's alternative courses of action. Since our analysis applies to the follower's optimization problem it may figure as a building block in a theory of oligopolistic competition in the supermarket industry.

This paper is organized as follows: §2 develops and discusses the buyer behavior model. §3 states the optimization problem, and interprets the optimality conditions. §4 presents a phase portrait analysis and the pattern of optimal policies. Moreover it summarizes the primary results and draws managerial consequences. An appendix contains some formal technical derivations. Detailed proofs are available from the authors on request.

2. The Model

We want to study the price and advertising policy that maximizes the retailer's cumulative discounted profit in the long run. As was argued in the introduction, current prices influence not only current sales but also the store price image. So they also have an impact on future sales. Therefore, we are led to regard the response model, on which we base the maximization, as a dynamic system. The overall structure of this system is as follows: The store's advertising and price image share in the role of drawing customers into the store. The effect of prices, however, is twofold: first they determine the sales receipts from the customers in the store, second they govern the evolution of the store price image. In the case of convenience items, the prices that maximize sales receipts in the short run will lead to an unfavorable image in the long run, since buyers accept higher prices for these goods for some time and shun an expensive store only in the future.

The system is developed in the following three subsections: in 2.1 our concept of the store price image is explained in detail and the dynamics of image formation are described. 2.2 outlines the sales response model. In a first stage the number of visitors to the store is determined as a function of the current price image and advertising. A second stage consists of a model which views the sales receipts per visitor as a function of actual prices. In 2.3 we discuss some aspects of interaction between the image dynamics and the response model.

2.1. *Price Image Formation*

The purchase behavior model to be outlined here could be derived from a detailed explanation of the consumer's learning and decision process, allowing for a heterogeneous population. To simplify exposition, however, we use as a model a consumer that represents the average of the total population.

Consumers form a price image for every store they visit. The price image R of a store is an aggregate measure of what consumers remember on how the prices p charged in that store compare to prices p^0 normally charged elsewhere. Let h denote a weighting scheme measuring the importance that consumers attribute to the various articles when they judge the price level of a store.[1] If a store's price level $h'p$ is usually higher than the normal level $h'p^0$, then it will acquire the image of being expensive. Formally, the store price image R can be conceived of as an average of the differences $h'(p^0 - p)$ remembered from the past.[2] This means that the price image R of an expensive store will be

[1] Note that p, p^0 and h are vectors that have a component for each item on the market. A prime on a vector indicates transposition. So the scalar product $h'p$ represents the price level of the store as a real index number.

[2] Usually store price images are conceived of as weighted averages of price perceptions, see e.g. Brown (1969) or Gabor (1977). Other contributions, however, emphasize that they result from comparisons.

negative. One might adopt adaptive learning as an appropriate averaging concept. This would lead to the following dynamic equation for the store price image:

$$\dot{R} = \gamma[h'(p^0 - p) - R], \tag{1}$$

in which $\dot{R}$ denotes the time derivative of R and the positive constant γ denotes the learning rate. Note that p and R depend on time. As it stands, however, (1) is an oversimplified model of the dynamics of store price image. According to (1) the impact of a difference $h'(p^0 - p)$ would be the same, irrespective of how many people perceive it. So, as a more adequate formulation we propose

$$\dot{R} = \gamma[h'(p^0 - p)v - R]. \tag{2}$$

In (2) v denotes the number of shoppers visiting the store per time unit. As will be argued later v depends on the store's current price image R and advertising effort a. According to this formulation the information on actual prices enters our model weighted by the number of persons exposed to them. This feature distinguishes our approach from unweighted adaptive learning models, e.g. Schmalensee (1978), Kotowitz and Mathewson (1979), or Conrad (1985). Another weighted formulation has been proposed in Spremann (1985). He applies the sales rate as a weight instead of v. But this proposal is not satisfactory for our purpose either. Because, according to him, high prices act as a brake on the decline of the store price image since they depress the sales rate.

Some clarification is necessary concerning the weighting scheme h. The vector h has a component for every convenience item a consumer may buy. It would be rational from the consumer's point of view to choose these weights proportionally to the average quantities bought of the respective goods per week. In this case, the information that enters the store price image would indicate the potential weekly savings that would accrue from buying regularly at the shop in question, compared to buying at the reference price level. Consumer rationality in this sense would greatly simplify the problem of measuring store price images, since a retail price index from public statistics could serve as a proxy for $h'p^0$, and $vh'p$ would correspond to the store's monetary sales per period. Brown (1969) and Gabor (1977), however, present empirical evidence against this hypothesis. Basically, the sensitivity of consumer's assessment of a store's price level to the price of an article is represented by the article's h-weight. In general, price sensitivity is influenced by price advertising as should be the case with the h-weights. To simplify matters, however, we consider them data for the store's price and advertising policy.

Our model formally treats the reference prices as constant parameters. This may call for justification, especially as we say that the optimal equilibrium of the store price image actually oscillates due to oscillations in the environmental parameters. Of course, consumers are constantly receiving information on prices. So, not only the store price images but also the reference prices will constantly be updated. This updating process, however, is not substantially influenced by policy measures of a single supplier on the market, because the reference price of a convenience item emerges from the totality of information on prices for the respective product class, e.g. from price advertising by manufacturers and competing stores, from shopping experience, and from word-of-

---

Nyström (1970), for example, measures an individual's store price image as a fraction n^+/n where n^+ is the number of price comparisons favorable to the considered store among a total of n comparisons elicited. Eckhardt (1976) defines a store price image as a weighted sum of differences between a reference price and the actual price observed in the store. We adopt this concept but introduce the time dimension by considering image formation as an adaptive learning process. The adaptive learning approach has already been proposed in Nyström (1970).

mouth. In competitive markets the reference prices to judge a store's price level will always be made up of the prices charged or advertised by competitors. As stated in the introduction we consider only companies that do not plan to influence their competitor's policy. For such companies, however, the adaptation process of reference prices is exogenous, i.e. part of the environment. Clearly, decision makers are well aware of the fact that future changes will occur in the relevant environment. We assume, however, that information on these changes, especially on the future changes of reference prices, is not available in advance; no trend, no seasonal pattern or the like is identified in related time series. In principle, decision makers could consider alternative scenarios of future development. This would seem appropriate for strategic decisions but less so for current pricing and advertising. We choose to consider these in a planning framework that is deterministic, albeit dynamic. The best policy, then, is to adopt a rolling horizon planning philosophy, i.e. to proceed as if the parameters remained constant at a level that corresponds to the latest information, and revise as new knowledge becomes available. Of course, the assumption that no trends or seasonal patterns are known is a crude abstraction from reality. To account for them, however, would obscure the results rather than yield additional insight.

2.2. *Sales Response Model*

As outlined in the introduction, we consider two stages of the consumer purchase decision on convenience goods. The first stage is store choice; the second is the decision on the quantities of each article to buy. Our model represents store choice behavior by the number v of visitors to the store per time unit. We assume that v is a function of current store price image R and current advertising outlay a. Specifically, let

$$v = a^{\omega} f(R) \tag{3}$$

with constant advertising elasticity ω $(0 < \omega < 1)$ and a price image response function f with positive first derivative f_R.

According to (3) the interaction between price image and advertising is complementary. A favorable price image enhances the credibility of advertising, while heavy advertising (e.g. more outlets, better accessibility, billboards) gives more impact to a favorable price image. We suppose, however, that the marginal effect of both advertising and the price image is decreasing, and that the complementarity is weaker than the joint decrease in marginal effectiveness. Formally, we assume

$$v_{aa} v_{RR} - v_{aR}^2 > 0 \tag{4}$$

i.e. v is strictly concave jointly in a and R. This implies that the second derivative f_{RR} of f is negative everywhere: a fixed difference in R will have a greater impact on the visit frequency v when the current price image is unfavorable than when it is favorable. For extremely favorable values of the price image this seems immediately plausible. Duplication effects and exhaustion of customer potential will hinder further improvement. But $f_{RR} < 0$ should hold for unfavorable price images also. To see this, note that, according to (2), a larger negative difference $h'(p^0 - p)$ is needed to achieve a given negative value $\dot{R}$ when R is low than when it is at a high value. There are two reasons for this: first, for negative R the term $-\gamma R$ in (2) acts in the positive direction and, second, in this case v will be small, so that the negative information on the store's price level is not so widely recognized. So R must be measured in such a way that a given difference in R has a greater impact on f for low values of R than for high ones. For some negative level $\bar{R}$ the visit frequency will be nullified.

Now let us turn to the determination of sales quantity. We assume that price response of visitors in the store is linear. So

$$s = A(\bar{p} - p)v(a, R), \tag{5}$$

where s denotes the vector of sales rates, and $\bar{p}$ a vector of backstop prices, while A is a quadratic matrix of response coefficients a_{ij}.

The influence of the price for item j on the sales of i is measured by a_{ij}; $a_{ij} > 0$ means that a price cut for item j leads to an increase in the sales of i. The parameters of the response function are $\bar{p}$ and the a_{ij}. We assume $a_{ii} > 0$, and that A has symmetrically dominant diagonal in the sense of Selten (1970, p. 25), i.e. there exist positive numbers μ_i for every i such that

$$\sum_{j \neq i} \mu_j(|a_{ij}| + |a_{ji}|)/2 < \mu_i a_{ii}.$$

This can be interpreted as a dominance of direct price effects over indirect ones. As Selten has shown the above condition implies

$$A + A' \text{ is positive definite} \tag{6a}$$

and so

$$B = (A + A')^{-1} \tag{6b}$$

exists and is also positive definite.

2.3. *Interactions of Price Image Dynamics and Sales Response*

Having considered the sales response model, we can clarify further how the system dynamics (2) functions. If we had chosen (1) as system dynamics, advertising would have no influence on store price image formation. But according to (3) and (5), an improvement of the price image would increase advertising effectiveness. This fact would entail the following as a guideline for management: the more favorable the store price image, the more advertising is indicated. The underpinning of this guideline, however, would be weak, since it neglects the fact that advertising helps to improve the price image. We avoid this weakness by postulating (2) as the law of store price image formation.

According to our formulation, the store price image R is the only goodwill capital considered. In principle, advertising capital in the sense of Nerlove and Arrow (1962) could be included. But this would mean that advertising could create goodwill independently of customers' shopping experience. To include such image advertising would complicate the model and obscure the results. Since advertising by supermarkets is mainly price advertising whose claims can easily be verified, image advertising may be of minor importance.

One might ask whether advertising low prices will not override an image of having been expensive and why a company cannot improve its store price image instantaneously. Of course, we are not saying that a negative store price image makes consumers believe that they would not get the promoted items at advertised prices. But the more negative the store price image, the more consumers will be inclined to think that the low prices advertised are not representative of the store's price level in general and for the future. Price advertising usually covers only a minor portion of the items a consumer buys on a visit to the store. And if the store has the image of being expensive, consumers do not readily infer from advertised low prices that the nonpromoted items they want to buy will also be low-priced. Empirical evidence suggests that not even the special promotions consumers actually make use of induce them to improve the store's price image. Only a split-up of this image was achieved: one for special offers and one for regular staples (Eckhardt 1976, Diller 1981).

Furthermore, many consumers do not worry about the single purchase of convenience goods; rather they adapt buying habits in order to get the best value for their money and effort in the long run. The store price image mirrors these buying habits. Nyström (1970) emphasises that a store price image is an attitude rather than a perception. Attitudes are not changed at once because of advertising information. Price advertising by a supermarket may stimulate consumers to gain experience with the store, but only this experience will alter the store's price image.

3. Optimality Conditions

Let the constants r and c denote the discount rate and the vector of direct unit costs, respectively. Then, based on the model in §2, the problem is to find functions p and a of time t which maximize the present value

$$\int_0^\infty [(p(t) - c)'A(\bar{p} - p(t))v(a(t), R(t)) - a(t)]e^{-rt}dt \tag{7a}$$

subject to

$$\dot{R} = \gamma[h'(p^0 - p)v - R], \qquad R(0) = R_0 > \bar{R}, \qquad \bar{R} < 0. \tag{7b}$$

To solve (7), we use the maximum principle (see, for example, Kamien and Schwartz 1981). Define gross profit per visitor as

$$\phi(p) = (p - c)'A(\bar{p} - p) \qquad \text{and} \tag{8}$$

$$g(p, \lambda) = \phi(p) + \gamma\lambda h'(p^0 - p). \tag{9}$$

Then the current value Hamiltonian may be written as $H = g(p, \lambda)v(a, R) - a - \lambda\gamma R$ where λ denotes the shadow price of the price image R. The shadow price $\lambda(t)$ represents the per unit increase in the present value of the optimal profits after t, due to a small increase of R at time t. Similarly $g(p(t), \lambda(t))$ is the value of one additional visitor at time t. It is composed of that visitor's expected contribution, $\phi(p(t))$ to current gross profit, and $\lambda\gamma h'(p^0 - p)$ to future profits. The latter contribution is expressed as the change in price image, induced by the additional visitor valued at the shadow price λ.

The necessary optimality conditions are:

$$H_p = g_p v = 0, \qquad \text{so that} \qquad g_p = 0, \tag{10}$$

$$H_a = gv_a - 1 = 0, \qquad \text{so that} \qquad a^{1-\omega} = g\omega f(R), \tag{11}$$

$$\dot{\lambda} = r\lambda - H_R = (r + \gamma)\lambda - gv_R. \tag{12}$$

(10) may be rewritten as

$$p = (A + A')^{-1}(A\bar{p} + A'c - \gamma\lambda h) = p^* - \gamma\lambda Bh \tag{13}$$

with B as defined in (6b). In (13)

$$p^* = B(A\bar{p} + A'c) \tag{14}$$

is the myopic monopolist's price vector; it maximizes gross profit per visitor $\phi(p)$. Note that ϕ is a strictly concave function due to (6a). Therefore the level set $\{p \mid \phi(p) \geq 0\}$ is a convex set in $\mathbb{R}^n$. Denote the unique intersection of the boundary of this set with the ray $\{p \mid p = p^* - \gamma\lambda Bh, \lambda \geq 0\}$ by p^c. We now assume that

$$h'p^c < h'p^0 < h'p^*. \tag{15}$$

The first inequality in (15) means that the price vector p^c leading to zero returns always compares favorably to the vector p^0 of reference prices from the consumer's point of view. The second one states that the myopically optimal price p^* would always pull down the price image. The case that p^0 does not belong to the set defined by (15) is of minor interest.

3.1. *Interpretation of Optimality Conditions*

We now interpret the necessary conditions as they relate to optimal price and advertising. From (11) it follows that $g > 0$. This fact and (12) imply that the stationary shadow price is positive. Furthermore, assuming $\lambda(t) \leq 0$ for some t and using (12) leads to a contradiction. So the shadow price λ of the store price image is always positive. If b_i denotes the ith row of B and if $b_i h < 0$ holds, then, by (13), the optimal price p_i for the ith item is higher than the myopic price p_i^*. On the contrary, $b_i h > 0$ leads to a p_i which is lower than p_i^*. In general, and on average according to the weights h, optimal retail prices will always be lower than the myopic monopoly prices p^* and decrease with the shadow price λ. Formally,

$$h' p_\lambda = -\gamma h' B h < 0 \qquad \text{and} \tag{16a}$$

$$h'(p - p^*) = -\gamma \lambda h' B h < 0 \tag{16b}$$

follow from (6b) and (13), since λ is positive. The intuitive reason for this is the long-run aftereffect of prices through the store price image.

Next (11) states that the last dollar spent for advertising should only just pay off. To the company it is worth the number of additional visits it triggers, valued at the worth g per visit. Note that $a = g\omega v$, according to (3) and (11). So optimal advertising outlay per visitor is $a/v = \omega g$, i.e. advertising elasticity times value of one additional visitor per time unit.

Inspection of the second derivatives of the Hamiltonian shows that the maximized Hamiltonian is concave in R. This, together with the limiting transversality condition, guarantees the sufficiency of the optimality conditions.

4. Phase Portrait Analysis and Optimal Marketing Policies

Any particular solution of the canonical system (7b), (12) along with the Hamiltonian maximizing conditions (10), (11) determines R and λ as a function of time. For a given (R, λ), the solution (p, a) of (10), (11) is unique and independent of t, and so the further development of the time path is determined independently of t. Consequently, it will be possible to represent any particular solution to (7b), (12), (10), (11) as a curve in (R, λ) space. Such a representation is termed a phase portrait of (7b), (12), (10), (11) in (R, λ) space. It allows us to write λ as a function of R, and vice versa, at least locally. Such a function $\lambda(R)$ enables us, furthermore, to write the optimal controls p, a as functions of the state variable R alone. This follows at once from

LEMMA 1. (10), (11) *determine implicit functions* $p = p(R, \lambda)$ *and* $a = a(R, \lambda)$ *with partial derivatives*

$$p_R = 0, \tag{17a}$$

$$p_\lambda = -\gamma B h, \qquad \textit{and} \tag{17b}$$

$$a_R = -v_{aR}/v_{aa} > 0, \tag{17c}$$

$$a_\lambda = \gamma (p^0 - p)' h v \omega / (1 - \omega). \tag{17d}$$

The proof of Lemma 1 is a direct application of the implicit function theorem to equations (10), (11). From (17d) we conclude

$$a_\lambda \gtreqless 0 \qquad \text{if and only if} \qquad h' p^0 \gtreqless h' p.$$

Given any function $\lambda(R)$ according to the phase portrait, the derivatives of the optimal controls $p(R) = p(R, \lambda(R))$, $a(R) = a(R, \lambda(R))$ are

$$\frac{dp}{dR} = p_R + p_\lambda \frac{d\lambda}{dR}, \tag{18a}$$

$$\frac{da}{dR} = a_R + a_\lambda \frac{d\lambda}{dR}. \tag{18b}$$

They indicate, for any time instant t, the impact of a small disturbance in the price image R at t on optimal price and advertising at t. (18) shows this impact decomposed into two components: Recall that $\lambda(t)$ is the change in discounted future profits per unit of a small disturbance in $R(t)$. So $p_R(t)$ represents the change in optimal price at t per unit of the disturbance in $R(t)$, if that disturbance were not to influence future profits, i.e. if $d\lambda/dR = 0$. Consequently, p_R signifies the direct, or short-run, impact, while $p_\lambda d\lambda/dR$ adjusts for long-run considerations. Of course, a similar interpretation holds for advertising. Lemma 1 shows that there is no short-run impact of the store price image on current price, and that the direct impact of price image on advertising is positive. We shall reconsider the long-run adjustments when we know about $d\lambda/dR$.

Now we substitute the functions $p(R, \lambda)$, $a(R, \lambda)$ in the state equation (7b) and the costate equation (12) to obtain the canonical system. This is an autonomous system of two differential equations in R and λ. To study this system, we first determine the shape of the isoclines $\dot{R} = 0$, $\dot{\lambda} = 0$. As already pointed out in the discussion of (15), there exists a unique $\lambda > 0$, λ^c, say, such that $p^c = p^* - \gamma\lambda^c Bh$. A simple, but cumbersome analysis using this result establishes

LEMMA 2. *The $\dot{R} = 0$ locus increases monotonically and intersects the R-axis at $\tilde{R}$ with $\bar{R} < \tilde{R} < 0$. The $\dot{\lambda} = 0$ isocline is* U*-shaped, and has a unique pole at $\lambda = 0$. It attains its minimum at $\lambda = \tilde{\lambda} < \lambda^c$.*

We are now able to state the first primary result of this section. It identifies the saddle point path toward the stationary point as the unique optimal solution to the canonical system. Every other solution would eventually lead to negative current profit, due to prices below cost, or to negative λ which has been excluded in §3 above. Furthermore, it yields the information $d\lambda/dR < 0$, needed to determine the long-run adjustment terms in (18).

THEOREM 1. *There exists a unique stationary point $(\hat{R}, \hat{\lambda})$ of the canonical system, located in the region $\{(R, \lambda) | 0 \le \lambda \le \lambda^c\}$. This equilibrium is a saddle point with downward sloping stable path. Moreover, $\hat{\lambda} < \tilde{\lambda}$.*

A proof of Theorem 1 is sketched in the appendix. The (R, λ) phase diagram is illustrated in Figure 1. Economically speaking, the downward sloping saddle point path says that the price image behaves as a scarce resource: the lower the image, the higher its value. Our proof guarantees the existence of the stable path only in a neighborhood of

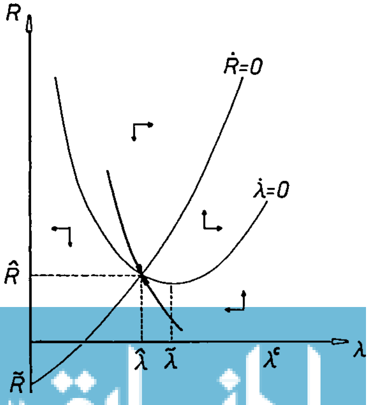


FIGURE 1. State-Costate Phase Portrait

the equilibrium. Using a global saddle point theorem, however (see, for example, Seierstad and Sydsaeter 1987, Theorem 3.10.2, or Hartman 1982, Theorem 1.2 in Chapter 8), the global existence of the saddle point path can be established.

In the remainder of this section, we study the feedback laws of the marketing instruments under optimal use. To this end, we first consider the (R, p_i) phase diagrams, where p_i denotes the price of the ith item. These diagrams can easily be obtained from the (R, λ) diagram because, according to (13), the price is a function of λ alone. There are two types of (R, p_i)-diagrams to be distinguished, depending on whether $b_i h$ is positive or negative. As we saw in §2.3, the first case is the usual one. The saddle point path is upward sloping in this case, and downward sloping in the other one. In the degenerate case $b_i h = 0$ the isocline $\dot{p} = 0$ is horizontal and coincides with the saddle point path.

A price index over all goods, formed according to the sensitivity weights h always behaves in the typical way (Figure 2a). Figure 2b represents the optimal pricing policy only for articles with low sensitivity weight h_i that are close substitutes for others with higher weight. No discount should be offered on such items as it would detract demand from the high sensitivity substitutes and the strongest possible effect on price image could not be attained.

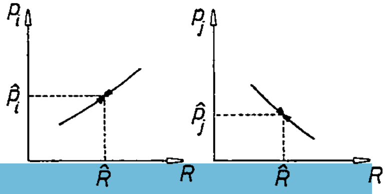
We now turn to the analysis of the (R, a) phase portrait. Whereas both in the (R, λ) and the (R, p_i) diagrams the monotonicity of the saddle point path holds in the entire interval $(\bar{R}, \infty)$, in the (R, a) plane this property turns out to be true only locally at the equilibrium point. According to Lemma 1 and Theorem 1 above, the optimal advertising path may be represented in the (R, a) plane in the form $a(R) = a(R, \lambda(R))$. It can be shown that $a(R)$ is locally increasing at the equilibrium, if either $\hat{R}$ is nonpositive or small, or if price image response f is not too sharply bent at the equilibrium $(\hat{R}, \hat{a})$.

THEOREM 2. *Let $\hat{a} = a(\hat{R}, \lambda(\hat{R}))$ denote the optimal equilibrium level of advertising. Then the slope of the saddle point path is positive in a neighborhood of $(\hat{R}, \hat{a})$ if either $\hat{R} \leq 0$ or the condition*

$$-f_{RR}R/f_R < 2(r/\gamma + \omega) + 1 \tag{19}$$

holds at the equilibrium.

The proof of this theorem is outlined in the appendix; $-f_{RR}R/f_R$ is the elasticity of marginal response to price image, i.e. the percent change in marginal response for a 1% change in R. If this is not too large in the sense of (19), then the slope of the optimal path will be positive near the equilibrium. Furthermore, the proof (see (A.7), appendix) shows that this slope will also be positive, when the optimal equilibrium level of R is rather low, i.e. if it is not worthwhile for the company to invest much in its price image. The only situation in which the saddle point path is downward sloping (i.e. in which optimal advertising is relatively large during a phase in which the price image is built



(a) Typical Item i. (b) Atypical Item j.

FIGURE 2. Stable Path in the Price Image-Price Phase Plane

up) is when the elasticity of marginal response to price image is large, i.e. when f is sharply bent, and prices are low, at the equilibrium.

REMARK. Condition (19) is satisfied for functions of the form $f(R) = \text{const.} (R - \bar{R})^\beta$ with $0 < \beta < 1 - \omega$, as well as for log-type functions f. Also a condition merely on v such as $-v_{RR}R/v_R < 2\omega + 1$ is sufficient for an upward sloping optimal path.

The optimal behavior of the advertising rate is shown in Figure 3.

The results on the optimal advertising pattern allow for the following interpretation. First consider a positive equilibrium level $\hat{R}$ of the store price image. It occurs when the company serves primarily price-sensitive regular customers, and cannot capitalize on a decisive locational advantage nor on strong passing trade that does not consider a store price image before deciding to enter the respective store. For positive $\hat{R}$, Theorem 2 shows that the optimal path in the (R, a) diagram is falling (rising) near the equilibrium, if a further increase in price image does (does not) lead to a rapid decrease in the marginal effect on visits.

According to the remark above, we think that condition (19) usually holds true, so that the optimal advertising rate is relatively high as long as price image is above equilibrium. In phases where image is below equilibrium, optimal advertising, however, is lower than when image is above equilibrium. But the less frequent case in which (19) does not hold cannot entirely be ruled out. It may occur especially when some kind of threshold phenomenon determines buyers' response to price image near the equilibrium.

Now consider the case of a negative equilibrium level of the price image. It occurs when customers buy at the store, although they expect that prices are high, e.g. in a tourist area, when time for shopping is short, or when there is no low-priced competition within reach. Under these conditions, Theorem 2 states that advertising is always higher for a current price image above equilibrium, than in a situation where price image has to be improved.

Note that Theorem 2 is a local result only. As sketched in Figure 3, the saddle point path in the (R, a) phase portrait is not monotonic for low levels of the price image. Although this behavior is not proved in Theorem 2, it has been established by numerical investigations, as well as geometrical considerations. It means that if price image is far below equilibrium, a drop in price image would increase the profitability of advertising. Presumably, it is an exception when a company reaches this situation.

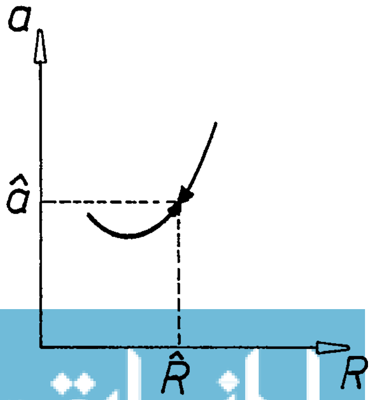


FIGURE 3. Stable Path in the Price Image-Advertising Plane.

(18b) means that in the case of a rising optimal path in the (R, a) plane, the short-run impact of advertising dominates over the long-run effect; while for a falling path the reverse is true. The short-run impact is to back up price image to attract visitors and thus enhance current sales, while the long-run effect is to accelerate price image formation for better future sales.

Let us consider the managerial consequences of Theorem 2 for the usual case of a rising saddle point path in the advertising/price image diagram.

There are two situations:

1° If the store price image is *above* equilibrium, the store management can be somewhat less concerned about price image, and refrain from strong price incentives to buyers. Advertising, however, should be even more intensive than at equilibrium. In this situation, the company capitalizes on an advertising effectiveness which is enhanced by a favorable price image.

2° If the store price image has declined *below* equilibrium, the store management should intensify price incentives by special offers or a general price reduction. Overall advertising budget, however, should not be boosted above its equilibrium level. Of course, price reductions may reasonably be backed up by price advertising, but this should be financed by reducing general advertising. The reason is that advertising effectiveness is handicapped by a damaged price image. In the case of Figure 3 the U-shaped part of the saddle point path might refer to a situation where the price advertising needed cannot be financed by reducing other kinds of advertising.

In effect, this means that advertising level will be increased only after the price image has been improved. This rule is somewhat similar to the well-known "consistency rule" in strategic planning (see Farris and Reibstein 1979). Within a static framework this rule holds only if high prices create high advertising effectiveness or if advertising reduces sensitivity to price. Although this is the classical hypothesis, most of the empirical evidence available on the issue (e.g. Prasad and Ring 1976, Wittink 1977, Woodside and Waddle 1975) indicate the reverse relationship. So, for the tactical policy problem considered above, it may seem a surprising result that a "consistency rule" follows from the assumption that advertising increases sensitivity to price. Our analysis reconciles this assumption and the "consistency rule" in a dynamic context by viewing both price and advertising as relative to an optimal steady state level. The price effect that interacts with advertising is the effect of the price image in our model: heavy advertising is optimal when the price image is favorable, i.e. when the store's perceived price level is low. But just then current optimal prices are relatively high. The dynamic response hypothesis reverses the relation between current prices and advertising, so that complementary interaction causes a "consistency rule" to be optimal.

5. Conclusion

We have considered a dynamic model of the interaction between prices and advertising on convenience goods purchasing behavior. The influence of prices is twofold. In the short run, prices influence the quantity each shopper buys when (s)he visits the store once. Sales quantity in this sense will be rather inelastic with respect to prices, so that fairly high prices would seem optimal. But there is another effect of prices in the long run: If customers observe high prices in a store, then a reputation for being expensive will gradually be acquired. Taking this into account will lead to prices considerably lower than the short-run optimal ones. We have assumed that advertising and a favorable reputation concerning value for money will interact in a complementary way to attract customers to the store. Our analysis suggests that, usually, optimal advertising is high when the price level is. So we arrive at a guideline that is quite similar to the one derived from the oversimplified dynamics (1), as discussed in §2.3. The key assumption, on which the result rests, is the complementary interaction of reputation and advertising.

As directions for future research, we consider the following:

—Examining other modes of interaction between the store price image and advertising.

—Sensitivity analyses of the equilibrium might be able to back up managerial intuition for assessing whether actual price image is below or above equilibrium.

—Generalizing the concept of reputation to other aspects of the service quality a store offers, such as variety, quality, and atmosphere. In this model, more marketing instruments will have to be introduced, and adequate interaction hypotheses developed that lead to a one-dimensional global measure of reputation, and shed additional light on the problem of marketing-mix interaction.

—Oligopolistic situations may be analyzed by considering explicitly the influence of competitors on the reference prices.[3]

Acknowledgement. This work was partly supported by the Austrian Science Foundation (project no. S 3204). We are grateful to Alois Steindl, Vienna, for carrying out many numerical experiments that were essential in guiding us towards the results presented here. Furthermore we would like to mention Klaus Spremann, Ulm, whose 1985 article inspired us.

[3] This paper was received in October 1985 and has been with the authors 8 months for 2 revisions.

Appendix

Proof of Theorem 1[4]

First, we have to show that there is an intersection of the $\dot{R} = 0$ and the $\dot{\lambda} = 0$ loci. To this end we observe that by Lemma 2 the isocline $\dot{\lambda} = 0$ has a pole at $\lambda = 0$, whereas $\dot{R} = 0$ is finite for $\lambda = 0$. Thus, it suffices to show that for $\lambda = \lambda^c$ the $\dot{R} = 0$ curve is above the $\dot{\lambda} = 0$ locus. This is a technical matter which we omit here.

The elements of the Jacobian matrix of the canonical system (7b), (12) are

$$\dot{R}_R = \gamma(p^0 - p)'hv_R/(1-\omega) - \gamma, \tag{A.1}$$

$$\dot{R} = \gamma^2 h'Bhv + [\gamma(p^0 - p)'h\omega v]^2/[a(1-\omega)] > 0, \tag{A.2}$$

$$\begin{aligned}\dot{\lambda}_R &= -(v_{RR} + v_{aR}a_R)g \\ &= -(v_{aa}v_{RR} - v_{aR}^2)a^{1-\omega}/[\omega f(R)v_{aa}] > 0,\end{aligned} \tag{A.3}$$

$$\bar{\lambda}_\lambda = r + \gamma - \gamma h'(p^0 - p)v_R/(1-\omega) = r - \dot{R}_R. \tag{A.4}$$

The signs of these can be derived from the following properties, which are immediate consequences of (3) and (4):

$$\begin{aligned} &v_a = \omega v/a > 0, \qquad v_{aa} = (\omega - 1)v_a/a < 0, \\ &v_R = v f_R/f > 0, \qquad v_{RR} = v_R f_{RR}/f_R < 0, \\ &v_{aR} = v_{aR}/f = \omega v_R/a > 0. \end{aligned} \tag{A.5}$$

Along the locus $\dot{R} = 0$ we have

$$\dot{R}_R < 0 \qquad \text{and} \qquad \dot{\lambda}_\lambda > 0. \tag{A.6}$$

From (A.3) and (A.6) we conclude that the slope of the $\dot{\lambda} = 0$ curve is negative at $R = \hat{R}$, i.e.

$$\left.\frac{dR}{d\lambda}\right|_{\substack{\dot{\lambda}=0 \\ R=\hat{R}}} = -[\dot{\lambda}_\lambda/\dot{\lambda}_R]|_{R=\hat{R}} < 0.$$

So the stationary point $(\hat{R}, \hat{\lambda})$ is always in the region where the $\dot{\lambda} = 0$ locus is falling, i.e. $\hat{\lambda} < \tilde{\lambda}$. This proves the uniqueness of the equilibrium since the $\dot{R} = 0$ locus is increasing. (A.4) allows the Jacobian determinant of the canonical system to be written as

$$\det J = -\dot{R}\dot{\bar{\lambda}}_R + r\dot{R}_R - \dot{R}_\lambda\dot{\lambda}_R.$$

[4] The saddle point property of the equilibrium can also be derived from more general stability results, see e.g. Rockafellar (1976), Feinstein and Oren (1983). The monotonicity result, however, requires examination of the Jacobian.

According to (A.2), (A.3), and (A.6), det J, evaluated at the stationary point, is negative. This shows the saddle point property of $(\hat{R}, \hat{\lambda})$.

Obviously, the stable path converging to the saddle point is downward sloping (see Figure 1; the directions of the arrows follow from (A.2) and (A.3) and ensure the monotonicity of the saddle point path). ■

Proof of Theorem 2

Recall (18b), i.e.

$$\frac{da}{dR} = a_R + a_\lambda \frac{d\lambda}{dR}.$$

Consider first the case $\dot{R} \le 0$. From (7b) we see that $h'(p^0 - p) \le 0$. Then $a_\lambda \le 0$ follows from (17d) and $d\lambda/dR < 0$ from Theorem 1. Since a_R is positive (see (17c)), the same is true for da/dR.

For the other case $\dot{R} > 0$ consider the identity $\dot{\lambda} = \dot{R} d\lambda/dR$ in which $d\lambda/dR$ is a function of R while both $\dot{\lambda}$ and $\dot{R}$ are functions of R and $\lambda(R)$. So, differentiating totally with respect to R yields

$$\dot{\lambda}_R + \dot{\lambda}_\lambda \frac{d\lambda}{dR} = \frac{d^2\lambda}{dR^2}\dot{R} + \frac{d\lambda}{dR}\left(\dot{R}_R + \dot{R}_\lambda \frac{d\lambda}{dR}\right).$$

At the equilibrium, the first term on the right-hand side vanishes, and, abbreviating $b = \dot{\lambda}_\lambda - \dot{R}_R$, the remaining quadratic equation yields the solution

$$\frac{d\lambda}{dR} = [b \pm (b^2 + 4\dot{R}_\lambda \dot{\lambda}_R)^{1/2}]/(2\dot{R}_\lambda).$$

Since $d\lambda/dR$ must be negative, the positive root can be neglected. So

$$\frac{da}{dR} = a_R + \frac{a_\lambda b}{2\dot{R}_\lambda} - \left[\frac{a_\lambda^2 b^2}{4\dot{R}_\lambda^2} + \frac{\dot{\lambda}_R a_\lambda^2}{\dot{R}_\lambda}\right]^{1/2}.$$

Substituting (17), (A.2), and (A.3) for the derivatives, we obtain by straight-forward calculation that da/dR is positive if and only if

$$-\gamma R f_{RR}/f_R < 2(1 - Rf_R/f) + r/\gamma + (1 + r/\gamma)\frac{h'(p^* - p)}{h'(p^0 - p)} \qquad \text{(A.7)}$$

holds. From (A.5) we can conclude $\omega < 1 - Rf_R/f$. Moreover, $\dot{R} > 0$ yields $h'(p^0 - p) > 0$. These inequalities, along with the assumption (15), lead to

$$1 < \frac{h'(p^* - p)}{h'(p^0 - p)}.$$

Consequently, the left-hand side in (A.7) is greater than $2(r/\gamma + \omega) + 1$. From this follows (19) as a sufficient condition for an upward sloping saddle point path (see Figure 3). ■

References

Brown, F. E. (1969), "Price Image Versus Price Reality," *Journal of Marketing Research*, 6, 185–191.

Conrad, Klaus (1985), "Quality, Advertising, and the Formation of Goowill Under Dynamic Conditions," in Gustav Fleichtinger (Ed.), *Optimal Control Theory and Economic Analysis* 2, Amsterdam: North Holland.

Diller, Hermann (1981), "Die Variation des Preisniveaus von Sonderangeboten im Lebensmittel-Einzelhandel, Eine empirische Wirkungsanalyse," *Discussion Papers in Marketing* No. 1, Hochschule der Bundeswehr, Hamburg.

Eckhardt, Klaus (1976), *Sonderangebotspolitik in Warenhandelsbetrieben—Eine empirische Studie*, Wiesbaden: Gabler.

Farris, Paul W. and David J. Reibstein (1979), "How Prices, Ad Expenditures, and Profits are Linked," *Harvard Business Review*, 57 (November–December), 173–185.

Feinstein, C. D. and S. S. Oren (1983), "Local Stability Properties of the Modified Hamiltonian Dynamic System," *Journal of Economic Dynamics and Control*, 6, 386–394.

Gabor, André (1977), *Pricing: Principles and Practice*, London: Heinemann.

Hartman, Philip (1982), *Ordinary Differential Equations*. 2nd ed., Boston: Birkhäuser.

Kamien, Morton I. and Nancy L. Schwartz (1981), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, New York: North Holland.

Kotowitz, Yehuda and Frank Mathewson (1979), "Advertising, Consumer Information, and Product Quality," *The Bell Journal of Economics*, 10, 566–588.

Lilien, Gary L. and Philip Kotler (1983), *Marketing Decision Making—A Model-Building Approach*, New York: Harper & Row.

Nerlove, Mark and Kenneth J. Arrow (1962), "Optimal Advertising Policy Under Dynamic Conditions," *Econometrica*, 29, 129–142.

Nyström, Harry (1970), *Retail Pricing—An Integrated Economic and Psychological Approach*, Stockholm: Economic Research Institute at the Stockholm School of Economics.

Prasad, V. Kanti and L. Winston Ring (1976), "Measuring Sales Effects of Some Marketing Mix Variables and Their Interactions," *Journal of Marketing Research*, 14, 391–396.

Rockafellar, R. Tyrrell (1976), "Saddle Points of Hamiltonian Systems in Convex Problems Having a Non-Zero Discount Rate," in D. Cass and K. Shell (Eds.), *The Hamiltonian Approach to Dynamic Economics*, New York: Academic Press.

Schmalensee, Richard (1978), "A Model of Advertising and Product Quality," *Journal of Political Economy*, 86, 485–503.

Seierstad, Knut and Atle Sydsaeter (1987), *Optimal Control Theory for Economists*, Amsterdam: North Holland.

Selten, Reinhard (1970), *Preispolitik der Mehrproduktenunternehmung in der statischen Theorie*, Berlin: Springer.

Spremann, Klaus (1985), "The Signaling of Quality by Reputation," in Gustav Feichtinger (Ed.), *Optimal Control Theory and Economic Analysis* 2, Amsterdam: North Holland.

Wittink, Dick R. (1977), "Advertising Increases Sensitivity to Price," *Journal of Advertising Research*, 17 (April), 39–42.

Woodside, Arch G. and Gerald L. Waddle (1975), "Sales Effects of In-Store Advertising," *Journal of Advertising Research*, 15 (June), 29–33.